Optimal Control in Time-Varying Velocity Fields using α-Shapes

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7/7/14

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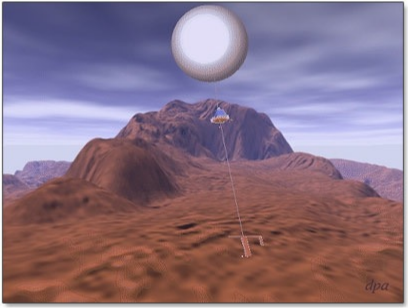
Problem Statement

- For a time-varying velocity field, plot an optimal course through the field for a vehicle controlled by speed and heading.
- Given:
  - Velocity field $\vec{U}(\vec{x}, t)$
  - Initial position $\vec{p}_0$, final position $\vec{p}_f$
  - Vehicle described by speed s and heading θ
- Find:
  - Optimal path $\mathcal{P}^*$ from $\vec{p}_0$ to $\vec{p}_f$ that minimizes some cost $J(\mathcal{P})$
  - Equivalently, the controls $s(t)$ and $\theta(t)$ to traverse $\mathcal{P}^*$

Applications

- Energy-efficient paths for commercial shipping and airliners
- Weakly-propelled atmospheric gliders and balloons
- Long-term research platforms in Earth's oceans and atmospheres



Rendering of a weakly propelled balloon exploring Venus

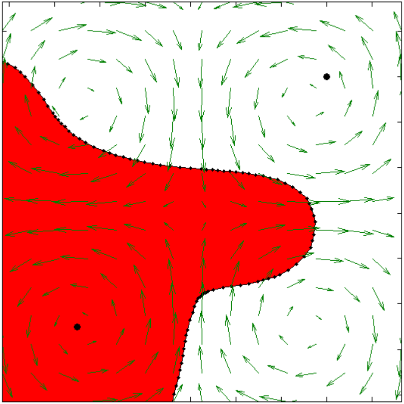
Image: *Global Aerospace Corporation*

Existing Methods

- Especially difficult when the vehicle's speed is small relative to the surrounding flow $s < |\vec{U}|$
  - Methods based on Dijkstra's shortest path on a grid fail
- Iterative relaxation, shooting methods, and piecewise optimizations are susceptible to local minima [*Rantzer et al.*, *Betts*, *Kruger et al.*]
- Recent efforts in set-based methods have proven highly effective [*Rhoads et al.*, *Lolla et al.*]

Front Propagation

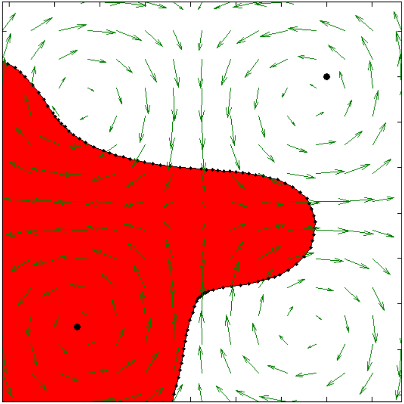
- Calculate the set $\mathcal{S}$ reachable by the vehicle at any time t



Expanding front in an alternating vortex flow

Front Propagation

- Calculate the set $\mathcal{S}$ reachable by the vehicle at any time t
- Expand the set according to a *front propagation rule*, derived from optimal control theory



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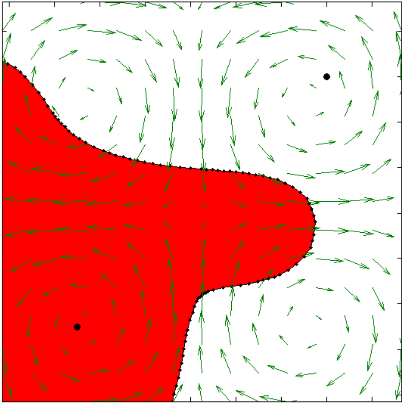
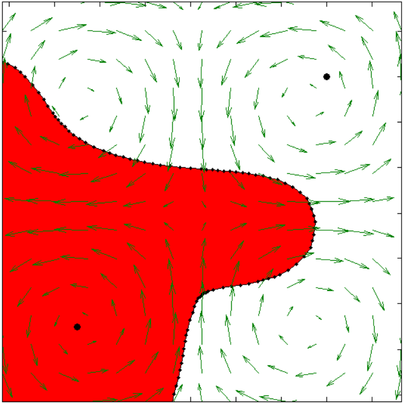
- Each point on the set boundary maps to an initial control value



Expanding front in an alternating vortex flow

- Calculate the set $\mathcal{S}$ reachable by the vehicle at any time t
- Expand the set according to a *front propagation rule*, derived from optimal control theory
- Each point on the set boundary maps to an initial control value
- Once $\mathcal{S}$ contains the target $\vec{p}_f$, the optimal solution $\mathcal{P}^*$ can be generated



Expanding front in an alternating vortex flow

Front Expansion Rule

- 2D time-optimal front propagation rule for a front element (vehicle) with position x, y and orientation θ

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} u + s\cos\theta \\ v + s\sin\theta \\ -u_y\cos^2\theta + (u_x - v_y)\sin\theta\cos\theta + v_x\sin^2\theta \end{bmatrix}$$

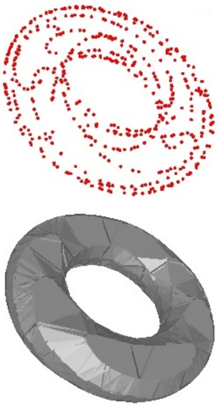
- $\dot{\theta}$ is derived from Pontryagin's Minimal Principle
- In the time-optimal problem $s^*(t) = s_{max}$

Extending to 3D

- Can we do the same in 3D?
- Meshing the boundary of the set in 3D is much more challenging
- Meshing scheme for surface manifold must support:
  - Expansion of reachable set from $t \to t + \Delta t$
  - Interpolation along the surface between iterations
  - *Trimming* of self-intersecting surface sections
- Use α**-shapes** to sidestep these issues

α-shapes

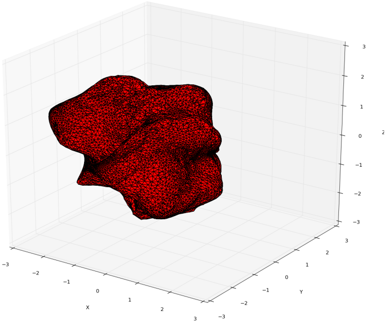
- Formalizes the idea of a *concave hull* of a set of points
- A length parameter α defines the minimum size of a concave exclusion
- The surface mesh of an α-shape can be computed from a point cloud n points in $\mathcal{O}(n^2)$ time



A 3D α-shape

Image: *CGAL Geometry Library*

- α-shapes allow us to work with a point cloud, meshing at the end of each iteration and sidestepping the previous requirements
- There is an approximation error associated with α-shapes, but this is similar to errors already typically accounted for
- This permits direct extension to time-optimal 3D problems!



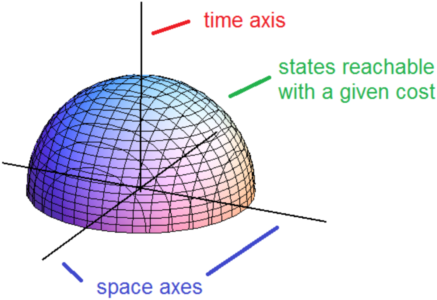
α-shape for a solution in the ABC flow

Mixed Time-Energy Cost Metrics

- In 2D, mixed time-energy optimality is much more practical than time alone

$$J(\mathcal{P}) = \int_0^{t_f} \rho_t + \rho_e s(t)^3 \, \mathrm{d}t$$

- $s(t)$ is no longer constant, so normal 2D method cannot approach this problem
- We can solve it by mapping it to a 3D problem and using α-shapes!



Mixed Time-Energy Cost Metrics

- The corresponding necessary condition for optimality is:

$$\dot{\theta} = -u_y \cos^2\theta + (u_x - v_y)\sin\theta\cos\theta + v_x \sin^2\theta$$
$$\dot{s} = \frac{1}{2}s\left(u_x \cos^2\theta + (u_y + v_x)\sin\theta\cos\theta + v_y \sin^2\theta\right)$$

- The distance traveled by a surface element in each ΔJ step can be found by

$$\Delta t = \frac{\Delta J}{\rho_e s^3 + \rho_t}$$

Conclusions

- α-shapes permit 3D front-propagation methods for optimal trajectories in velocity velocity fields
- Time optimal controls in 3D follow directly, given this technique
- Mixed time-energy optimization in 2D is possible by mapping the problem to 3D

Future Work

- What are the numerical effects of the choices of α and the interpolation threshold?
- Can this method be extended to the state space of other dynamical systems?
- Do any of these strategies apply when the velocity field is not fully known *a priori*?

Selected Literature

B. Rhoads, I. Mezić, and A. C. Poje, "Minimum time heading control of underpowered vehicles in time-varying ocean currents," *Ocean Engineering*, vol. 66, pp. 12–31, 2013.

T. Lolla, M. Ueckermann, K. Yigit, P. J. Haley, and P. F. Lermusiaux, "Path planning in time dependent flow fields using level set methods," in *Robotics and Automation (ICRA), 2012 IEEE International Conference on*, pp. 166–173, IEEE, 2012.

H. Edelsbrunner and E. P. Mücke, "Three-dimensional alpha shapes," *ACM Transactions on Graphics (TOG)*, vol. 13, no. 1, pp. 43–72, 1994.

M. Branicki and S. Wiggins, "An adaptive method for computing invariant manifolds in non-autonomous, three-dimensional dynamical systems," *Physica D: Nonlinear Phenomena*, vol. 238, no. 16, pp. 1625–1657, 2009.

"CGAL, Computational Geometry Algorithms Library."