Optimal Control in Time-Varying Velocity Fields using \( \alpha \)-Shapes

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Problem Statement

- For a time-varying velocity field, plot an optimal course through the field for a vehicle controlled by speed and heading.

- Given:
  - Velocity field $\vec{U}(\vec{x}, t)$
  - Initial position $\vec{p}_0$, final position $\vec{p}_f$
  - Vehicle described by speed $s$ and heading $\theta$

- Find:
  - Optimal path $\mathcal{P}^*$ from $\vec{p}_0$ to $\vec{p}_f$ that minimizes some cost $J(\mathcal{P})$
  - Equivalently, the controls $s(t)$ and $\theta(t)$ to traverse $\mathcal{P}^*$
Applications

- Energy-efficient paths for commercial shipping and airliners
- Weakly-propelled atmospheric gliders and balloons
- Long-term research platforms in Earth’s oceans and atmospheres

Rendering of a weakly propelled balloon exploring Venus

Image: Global Aerospace Corporation
Existing Methods

- Especially difficult when the vehicle’s speed is small relative to the surrounding flow $s < |\vec{U}|$
  - Methods based on Dijkstra’s shortest path on a grid fail

- Iterative relaxation, shooting methods, and piecewise optimizations are susceptible to local minima [Rantzer et al., Betts, Kruger et al.]

- Recent efforts in set-based methods have proven highly effective [Rhoads et al., Lolla et al.]
Front Propagation

- Calculate the set $S$ reachable by the vehicle at any time $t$

Expanding front in an alternating vortex flow
Front Propagation

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- Expand the set according to a front propagation rule, derived from optimal control theory

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Front Propagation

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- Expand the set according to a *front propagation rule*, derived from optimal control theory

- Each point on the set boundary maps to an initial control value

- Once $S$ contains the target $\vec{p}_f$, the optimal solution $\mathcal{P}^*$ can be generated

Expanding front in an alternating vortex flow
Front Expansion Rule

- 2D time-optimal front propagation rule for a front element (vehicle) with position $x, y$ and orientation $\theta$

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
u + s \cos \theta \\
v + s \sin \theta \\
-u_y \cos^2 \theta + (u_x - v_y) \sin \theta \cos \theta + v_x \sin^2 \theta
\end{bmatrix}
\]

- $\dot{\theta}$ is derived from Pontryagin’s Minimal Principle
- In the time-optimal problem $s^*(t) = s_{max}$
Can we do the same in 3D?

Meshing the boundary of the set in 3D is much more challenging

Meshing scheme for surface manifold must support:

- Expansion of reachable set from $t \rightarrow t + \Delta t$
- Interpolation along the surface between iterations
- *Trimming* of self-intersecting surface sections

Use $\alpha$-shapes to sidestep these issues
\( \alpha \)-shapes

- Formalizes the idea of a concave hull of a set of points.
- A length parameter \( \alpha \) defines the minimum size of a concave exclusion.
- The surface mesh of an \( \alpha \)-shape can be computed from a point cloud \( n \) points in \( \mathcal{O}(n^2) \) time.

A 3D \( \alpha \)-shape

Image: CGAL Geometry Library
Optimal Solutions with $\alpha$-shapes

- $\alpha$-shapes allow us to work with a point cloud, meshing at the end of each iteration and sidestepping the previous requirements.

- There is an approximation error associated with $\alpha$-shapes, but this is similar to errors already typically accounted for.

- This permits direct extension to time-optimal 3D problems!

$\alpha$-shape for a solution in the ABC flow.
Mixed Time-Energy Cost Metrics

- In 2D, mixed time-energy optimality is much more practical than time alone

\[ J(\mathcal{P}) = \int_{0}^{t_f} \rho_t + \rho_e s(t)^3 \, dt \]

- \( s(t) \) is no longer constant, so normal 2D method cannot approach this problem

- We can solve it by mapping it to a 3D problem and using \( \alpha \)-shapes!
The corresponding necessary condition for optimality is:

\[ \dot{\theta} = -u_y \cos^2 \theta + (u_x - v_y) \sin \theta \cos \theta + v_x \sin^2 \theta \]

\[ \dot{s} = \frac{1}{2} s \left( u_x \cos^2 \theta + (u_y + v_x) \sin \theta \cos \theta + v_y \sin^2 \theta \right) \]

The distance traveled by a surface element in each $\Delta J$ step can be found by

\[ \Delta t = \frac{\Delta J}{\rho_e s^3 + \rho_t} \]
Conclusions

- α-shapes permit 3D front-propagation methods for optimal trajectories in velocity velocity fields
- Time optimal controls in 3D follow directly, given this technique
- Mixed time-energy optimization in 2D is possible by mapping the problem to 3D
Future Work

- What are the numerical effects of the choices of $\alpha$ and the interpolation threshold?

- Can this method be extended to the state space of other dynamical systems?

- Do any of these strategies apply when the velocity field is not fully known \textit{a priori}?
Selected Literature


“CGAL, Computational Geometry Algorithms Library.”