# Céa's Method Derivations for the Variational Shape Cuts Project

Nicholas Sharp (nsharp@cs.cmu.edu)

May 8, 2019

This document explicitly walks through the application of Céa's method for shape derivatives to the energies considered in *Variational Surface Cutting*. For a general introduction to shape optimization and Céa's method, see the separate tutorial document.

Here we consider Poisson-constrained shape optimization problems of the form

$$\begin{array}{rcll} \Delta u & = & -K & \text{on} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \tag{1}$$

where the domain of interest $\Omega$ is a subset a surface $\mathcal{M}$.

We will consider several different cost functions, denoted by $J(\Omega)$. Note that these cost functions *do not* all have the form $J(\Omega) = \int_\Omega j(u(x))dx$; this generality forces us take variations for each energy, rather than using the simple form presented in the tutorial document.

For each choice of $J(\Omega)$, we will do the hard work and derive the expression for the Lagrangian and the values of the Lagrange multipliers at a critical point; the remaining steps proceed as in the tutorial.

## 1 Dirichlet Energy

We begin by considering the Dirichlet energy

$$J_D(\Omega) := \int_\Omega |\nabla u|^2.$$

The corresponding Lagrangian is

$$\mathcal{L}(\Omega, u, p, \lambda) := \int_\Omega |\nabla u|^2 + \int_\Omega p(\Delta u - f) + \int_{\partial\Omega} \lambda u. \tag{2}$$

We now evaluate the partial derivatives of $\mathcal{L}(\Omega, u, p, \lambda)$ with respect to $u$, $p$, and $\lambda$.

First, recall that the derivative of Dirichlet energy in a direction $v$ is

$$\frac{\partial J(\Omega)}{\partial u}(v) = \frac{\partial}{\partial u}(v) \int_\Omega \nabla u \cdot \nabla u = 2\int_\Omega \nabla u \cdot \nabla v \tag{3}$$

$$= 2\int_\Omega v\Delta u + 2\int_{\partial\Omega} v \frac{\partial u}{\partial n} \tag{4}$$

where the second line follows from Green's identities. Note that the signs in this expression depend on the sign convention for the Laplacian– we will always use the positive definite Laplace operator common in geometry.

Now, the derivative of $\mathcal{L}$ with respect to $u$ in a direction $v$ is given by

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial u}(v) &= \frac{\partial J(\Omega)}{\partial u}(v) + \int_{\Omega} p\Delta v + \int_{\partial\Omega} \lambda v \\
&= 2\int_{\Omega} v\Delta u + 2\int_{\partial\Omega} v\frac{\partial u}{\partial n} + \int_{\Omega} v\Delta p - \int_{\partial\Omega} p\frac{\partial v}{\partial n} + \int_{\partial\Omega} v\frac{\partial p}{\partial n} + \int_{\partial\Omega} \lambda v \qquad (5) \\
&= \int_{\Omega} (2v\Delta u + v\Delta p) + \int_{\partial\Omega} (2v\frac{\partial u}{\partial n} - p\frac{\partial v}{\partial n} + v\frac{\partial p}{\partial n} + \lambda v). \qquad (6)
\end{aligned}$$

where that we have also used the Poisson adjoint formula from the tutorial document.

The derivative of $\mathcal{L}$ with respect to $p$ in a direction $q$ is given by

$$\frac{\partial \mathcal{L}}{\partial p}(q) = \int_{\Omega} q(\Delta u - f). \qquad (7)$$

The derivative of $\mathcal{L}$ with respect to $\lambda$ in a direction $\mu$ is given by

$$\frac{\partial \mathcal{L}}{\partial \lambda}(\mu) = \int_{\partial\Omega} \mu u. \qquad (8)$$

At a critical point, we must have that the directional derivative vanishes along *all* directions, and thus each of the directional derivative expressions above must vanish for *all* choices of $(v, q, \mu)$. We can now consider some carefully chosen directions to determine the values of $(u, p, \lambda)$.

1. Consider $\frac{\partial \mathcal{L}}{\partial p}(q)$ for $q$ with $q = 0$ on $\partial\Omega$.

$$\frac{\partial \mathcal{L}}{\partial p}(q) = 0 = \int_{\Omega} q(\Delta u - f) \quad \Longrightarrow \quad \Delta u = f \quad \text{on} \quad \Omega.$$

2. Consider $\frac{\partial \mathcal{L}}{\partial \lambda}(\mu)$ for any $\mu$.

$$\frac{\partial \mathcal{L}}{\partial \lambda}(\mu) = 0 = \int_{\partial\Omega} \mu u \quad \Longrightarrow \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

3. Consider $\frac{\partial \mathcal{L}}{\partial u}(v)$ for $v$ with $v = 0$, $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$ (delta distributions on the interior are one possible choice)

$$\frac{\partial \mathcal{L}}{\partial u}(v) = 0 = \int_{\Omega} 2v\Delta u + v\Delta p + \cancelto{0}{\int_{\partial\Omega} 2v\frac{\partial u}{\partial n} - p\frac{\partial v}{\partial n} + v\frac{\partial p}{\partial n} + \lambda v} \quad \Longrightarrow \quad \Delta p = -2\Delta u$$

   Now that we know $\Delta p = -2\Delta u$, the first two integrals will cancel to 0 in all subsequent expressions.

4. Consider $\frac{\partial \mathcal{L}}{\partial u}(v)$ for $v$ with $v = 0$ on $\partial\Omega$

$$\frac{\partial \mathcal{L}}{\partial u}(v) = 0 = \int_{\Omega} \cancelto{0}{2v\Delta u + v\Delta p} + \int_{\partial\Omega} \cancelto{0}{2v\frac{\partial u}{\partial n}} - p\frac{\partial v}{\partial n} + \cancelto{0}{v\frac{\partial p}{\partial n} + \lambda v} \quad \Longrightarrow \quad p = 0 \quad \text{on} \quad \partial\Omega.$$

5. Consider $\frac{\partial \mathcal{L}}{\partial u}(v)$ for general $v$.

$$\frac{\partial \mathcal{L}}{\partial u}(v) = 0 = \int_{\Omega} \cancelto{0}{2v\Delta u + v\Delta p} + \int_{\partial\Omega} 2v\frac{\partial u}{\partial n} - \cancelto{0}{p\frac{\partial v}{\partial n}} + v\frac{\partial p}{\partial n} + \lambda v \quad \Longrightarrow \quad \lambda = -\frac{\partial p}{\partial n} - 2\frac{\partial u}{\partial n}$$

Note that because $\Delta p = -2\Delta u$ with identical Dirichlet-0 boundary conditions, $p = -2u$, and the problem is self-adjoint, allowing us to further simplify several of these expressions. Most significantly

$$\lambda = -\frac{\partial p}{\partial n} - 2\frac{\partial u}{\partial n} = 2\frac{\partial u}{\partial n} - 2\frac{\partial u}{\partial n} = 0.$$

In summary, for any fixed $\Omega$ the unique critical point in the other unknowns of the Lagrangian $\mathcal{L}(\Omega, u, p, \lambda)$ is given by

$$p = -2u \qquad \begin{aligned} \Delta u &= -K \quad \text{on} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \qquad \lambda = 0. \tag{9}$$

The shape derivative of the *unconstrained* Lagrangian at a critical point is then given by

$$D_\theta(\mathcal{L}) = D_\theta(J) = \int_{\partial\Omega} \left( |\nabla u|^2 + \lambda \frac{\partial u}{\partial n} \right) \theta \, ds = \int_{\partial\Omega} |\nabla u|^2 \theta \, ds = \int_{\partial\Omega} \left( \frac{\partial u}{\partial n} \right)^2 \theta \, ds.$$

## 2 Scaled Dirichlet Energy

We now consider the scaled Dirichlet energy

$$J_D(\Omega) := \int_\Omega a|\nabla u|^2.$$

The corresponding Lagrangian is

$$\mathcal{L}(\Omega, u, p, \lambda) := \int_\Omega a|\nabla u|^2 + \int_\Omega p(\Delta u - f) + \int_{\partial\Omega} \lambda u. \tag{10}$$

We now evaluate the partial derivatives of $\mathcal{L}(\Omega, u, p, \lambda)$ with respect to $u$, $p$, and $\lambda$.

The derivative of the scaled Dirichlet energy in a direction $v$ is

$$\frac{\partial J(\Omega)}{\partial u}(v) = \frac{\partial}{\partial u}(v) \int_\Omega a\nabla u \cdot \nabla u = 2\int_\Omega a\nabla u \cdot \nabla v \tag{11}$$

First, note that the following holds via traditional vector calculus for scalar fields $a, u, v$

$$a\nabla v \cdot \nabla u = \nabla \cdot (av\nabla u) + av\Delta u - v\nabla a \cdot \nabla u. \tag{12}$$

We can then apply this relationship along with Stokes' Theorem to see

$$\begin{aligned} \int_\Omega a\nabla v \cdot \nabla u &= \int_\Omega \nabla \cdot (av\nabla u) + av\Delta u - v\nabla a \cdot \nabla u \\ &= \int_{\partial\Omega} av\frac{\partial u}{\partial n} + \int_\Omega av\Delta u - v\nabla a \cdot \nabla u. \end{aligned} \tag{13}$$

This equivalence yields the derivative of the rescaled energy

$$\frac{\partial J(\Omega)}{\partial u} = \frac{\partial}{\partial u}\int_\Omega a\nabla u \cdot \nabla u = 2\int_\Omega a\nabla v \cdot \nabla u = 2\int_{\partial\Omega} av\frac{\partial u}{\partial n} + 2\int_\Omega av\Delta u - v\nabla a \cdot \nabla u. \tag{14}$$

Now, the derivative of $\mathcal{L}$ with respect to $u$ in a direction $v$ is given by

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial u}(v) &= \frac{\partial J(\Omega)}{\partial u}(v) + \int_\Omega p\Delta v + \int_{\partial\Omega} \lambda v \\
&= 2\int_\Omega av\Delta u - v\nabla a\cdot\nabla u + 2\int_{\partial\Omega} av\frac{\partial u}{\partial n} + \int_\Omega v\Delta p - \int_{\partial\Omega} p\frac{\partial v}{\partial n} + \int_{\partial\Omega} v\frac{\partial p}{\partial n} + \int_{\partial\Omega}\lambda v \qquad (15)\\
&= \int_\Omega (2av\Delta u - 2v\nabla a\cdot\nabla u + v\Delta p) + \int_{\partial\Omega} 2av\frac{\partial u}{\partial n} - p\frac{\partial v}{\partial n} + v\frac{\partial p}{\partial n} + \lambda v. \qquad (16)
\end{aligned}
$$

where that we have also used the Poisson adjoint formula from the tutorial document.

The derivative of $\mathcal{L}$ with respect to $p$ in a direction $q$ is given by

$$\frac{\partial \mathcal{L}}{\partial p}(q) = \int_\Omega q(\Delta u - f). \qquad (17)$$

The derivative of $\mathcal{L}$ with respect to $\lambda$ in a direction $\mu$ is given by

$$\frac{\partial \mathcal{L}}{\partial \lambda}(\mu) = \int_{\partial\Omega} \mu u. \qquad (18)$$

At a critical point, we must have that the directional derivative vanishes along *all* directions, and thus each of the directional derivative expressions above must vanish for *all* choices of $(v, q, \mu)$. We can now consider some carefully chosen directions to determine the values of $(u, p, \lambda)$.

1. Consider $\frac{\partial \mathcal{L}}{\partial p}(q)$ for $q$ with $q = 0$ on $\partial\Omega$.

$$\frac{\partial \mathcal{L}}{\partial p}(q) = 0 = \int_\Omega q(\Delta u - f) \quad \Longrightarrow \quad \Delta u = f \quad \text{on} \quad \Omega.$$

2. Consider $\frac{\partial \mathcal{L}}{\partial \lambda}(\mu)$ for any $\mu$.

$$\frac{\partial \mathcal{L}}{\partial \lambda}(\mu) = 0 = \int_{\partial\Omega} \mu u \quad \Longrightarrow \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

3. Consider $\frac{\partial \mathcal{L}}{\partial u}(v)$ for $v$ with $v = 0$, $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$ (delta distributions on the interior are one possible choice)

$$\frac{\partial \mathcal{L}}{\partial u}(v) = 0 = \int_\Omega (2av\Delta u - 2v\nabla a\cdot\nabla u + v\Delta p) + \cancelto{0}{\int_{\partial\Omega} 2av\frac{\partial u}{\partial n} - p\frac{\partial v}{\partial n} + v\frac{\partial p}{\partial n} + \lambda v} \quad \Longrightarrow \quad \Delta p = -2a\Delta u + 2\nabla a\cdot\nabla u$$

Now that we know $\Delta p = -2a\Delta u + 2\nabla a\cdot\nabla u$, the volume integral cancels to 0 in all subsequent expressions.

4. Consider $\frac{\partial \mathcal{L}}{\partial u}(v)$ for $v$ with $v = 0$ on $\partial\Omega$

$$\frac{\partial \mathcal{L}}{\partial u}(v) = 0 = \cancelto{0}{\int_\Omega (2av\Delta u - 2v\nabla a\cdot\nabla u + v\Delta p)} + \int_{\partial\Omega} \cancelto{0}{2av\frac{\partial u}{\partial n}} - p\frac{\partial v}{\partial n} + \cancelto{0}{v\frac{\partial p}{\partial n} + \lambda v} \quad \Longrightarrow \quad p = 0 \quad \text{on} \quad \partial\Omega.$$

5. Consider $\frac{\partial \mathcal{L}}{\partial u}(v)$ for general $v$.

$$\frac{\partial \mathcal{L}}{\partial u}(v) = 0 = \cancelto{0}{\int_\Omega (2av\Delta u - 2v\nabla a\cdot\nabla u + v\Delta p)} + \int_{\partial\Omega} 2av\frac{\partial u}{\partial n} - \cancelto{0}{p\frac{\partial v}{\partial n}} + v\frac{\partial p}{\partial n} + \lambda v \quad \Longrightarrow \quad \lambda = -\frac{\partial p}{\partial n} - 2a\frac{\partial u}{\partial n}$$

In summary, for any fixed $\Omega$ the unique critical point in the other unknowns of the Lagrangian $\mathcal{L}(\Omega, u, p, \lambda)$ is given by

$$\begin{aligned} \Delta p &= -2a\Delta u + 2\nabla a \cdot \nabla u \quad \text{on} \quad \Omega \\ p &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \qquad \begin{aligned} \Delta u &= -K \quad \text{on} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \qquad \lambda = -\frac{\partial p}{\partial n} - 2a\frac{\partial u}{\partial n}. \tag{19}$$

Note that the expression on the right hand side for $p$ is equivalently a scaled Laplacian

$$\Delta p = -2a\Delta u + 2\nabla a \cdot \nabla u = 2\nabla \cdot a\nabla u = -2\Delta_a u \tag{20}$$

The shape derivative of the *unconstrained* Lagrangian at a critical point is then given by

$$\begin{aligned} D_\theta(\mathcal{L}) = D_\theta(J) &= \int_{\partial\Omega} \left( a|\nabla u|^2 + \lambda \frac{\partial u}{\partial n} \right) \theta \; ds \\ &= \int_{\partial\Omega} \left( a|\nabla u|^2 - (\frac{\partial p}{\partial n} + 2a\frac{\partial u}{\partial n})\frac{\partial u}{\partial n} \right) \theta \; ds \\ &= \int_{\partial\Omega} \left( a\frac{\partial u}{\partial n}\frac{\partial u}{\partial n} - \frac{\partial p}{\partial n}\frac{\partial u}{\partial n} - 2a\frac{\partial u}{\partial n}\frac{\partial u}{\partial n} \right) \theta \; ds \\ &= \int_{\partial\Omega} \left( -a\frac{\partial u}{\partial n}\frac{\partial u}{\partial n} - \frac{\partial p}{\partial n}\frac{\partial u}{\partial n} \right) \theta \; ds \end{aligned} \tag{21}$$

As expected, if $a = 1$ everywhere this expression coincides with unscaled Dirichlet shape derivative.

# 3 Hencky Energy

We now consider the Hencky energy

$$J_D(\Omega) := \int_\Omega u^2.$$

Note that this energy has the simplest form described in the tutorial document, and the derivation there directly applies here without modification. For completeness, we will walk through the process in its entirety.

The corresponding Lagrangian is

$$\mathcal{L}(\Omega, u, p, \lambda) := \int_\Omega u^2 + \int_\Omega p(\Delta u - f) + \int_{\partial\Omega} \lambda u. \tag{22}$$

We now evaluate the partial derivatives of $\mathcal{L}(\Omega, u, p, \lambda)$ with respect to $u$, $p$, and $\lambda$.

The derivative of $\mathcal{L}$ with respect to $u$ in a direction $v$ is given by

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u}(v) &= \frac{\partial J(\Omega)}{\partial u}(v) + \int_\Omega p\Delta v + \int_{\partial\Omega} \lambda v \\ &= 2\int_\Omega vu + \int_\Omega v\Delta p - \int_{\partial\Omega} p\frac{\partial v}{\partial n} + \int_{\partial\Omega} v\frac{\partial p}{\partial n} + \int_{\partial\Omega} \lambda v \qquad (23) \\ &= \int_\Omega 2vu + v\Delta p + \int_{\partial\Omega} -p\frac{\partial v}{\partial n} + v\frac{\partial p}{\partial n} + \lambda v. \qquad (24) \end{aligned}$$

where that we have also used the Poisson adjoint formula from the tutorial document.

The derivative of $\mathcal{L}$ with respect to $p$ in a direction $q$ is given by

$$\frac{\partial \mathcal{L}}{\partial p}(q) = \int_\Omega q(\Delta u - f). \tag{25}$$

The derivative of $\mathcal{L}$ with respect to $\lambda$ in a direction $\mu$ is given by

$$\frac{\partial \mathcal{L}}{\partial \lambda}(\mu) = \int_{\partial\Omega} \mu u. \tag{26}$$

At a critical point, we must have that the directional derivative vanishes along *all* directions, and thus each of the directional derivative expressions above must vanish for *all* choices of $(v, q, \mu)$. We can now consider some carefully chosen directions to determine the values of $(u, p, \lambda)$.

1. Consider $\frac{\partial \mathcal{L}}{\partial p}(q)$ for $q$ with $q = 0$ on $\partial\Omega$.

$$\frac{\partial \mathcal{L}}{\partial p}(q) = 0 = \int_\Omega q(\Delta u - f) \quad \Longrightarrow \quad \Delta u = f \quad \text{on} \quad \Omega.$$

2. Consider $\frac{\partial \mathcal{L}}{\partial \lambda}(\mu)$ for any $\mu$.

$$\frac{\partial \mathcal{L}}{\partial \lambda}(\mu) = 0 = \int_{\partial\Omega} \mu u \quad \Longrightarrow \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

3. Consider $\frac{\partial \mathcal{L}}{\partial u}(v)$ for $v$ with $v = 0$, $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$ (delta distributions on the interior are one possible choice)

$$\frac{\partial \mathcal{L}}{\partial u}(v) = 0 = \int_\Omega 2vu + v\Delta p + \cancelto{0}{\int_{\partial\Omega} -p\frac{\partial v}{\partial n} + v\frac{\partial p}{\partial n} + \lambda v} \quad \Longrightarrow \quad \Delta p = -2u$$

   Now that we know $\Delta p = -2u$, the first two integrals will cancel to 0 in all subsequent expressions.

4. Consider $\frac{\partial \mathcal{L}}{\partial u}(v)$ for $v$ with $v = 0$ on $\partial\Omega$

$$\frac{\partial \mathcal{L}}{\partial u}(v) = 0 = \int_\Omega \cancelto{0}{2vu + v\Delta p} + \int_{\partial\Omega} -p\frac{\partial v}{\partial n} + \cancelto{0}{v\frac{\partial p}{\partial n} + \lambda v} \quad \Longrightarrow \quad p = 0 \quad \text{on} \quad \partial\Omega.$$

5. Consider $\frac{\partial \mathcal{L}}{\partial u}(v)$ for general $v$.

$$\frac{\partial \mathcal{L}}{\partial u}(v) = 0 = \int_\Omega \cancelto{0}{2v\Delta u + v\Delta p} + \int_{\partial\Omega} \cancelto{0}{-p\frac{\partial v}{\partial n}} + v\frac{\partial p}{\partial n} + \lambda v \quad \Longrightarrow \quad \lambda = -\frac{\partial p}{\partial n}$$

In summary, for any fixed $\Omega$ the unique critical point in the other unknowns of the Lagrangian $\mathcal{L}(\Omega, u, p, \lambda)$ is given by

$$\begin{array}{rclll} \Delta p &=& -2u & \text{on} & \Omega \\ p &=& 0 & \text{on} & \partial\Omega \end{array} \qquad \begin{array}{rclll} \Delta u &=& -K & \text{on} & \Omega \\ u &=& 0 & \text{on} & \partial\Omega \end{array} \qquad \lambda = -\frac{\partial p}{\partial n}. \tag{27}$$

The shape derivative of the *unconstrained* Lagrangian at a critical point is then given by

$$D_\theta(\mathcal{L}) = D_\theta(J) = \int_{\partial\Omega} \left(u^2 + \lambda \frac{\partial u}{\partial n}\right)\theta \, ds = \int_{\partial\Omega} \left(u^2 - \frac{\partial p}{\partial n}\frac{\partial u}{\partial n}\right)^2 \theta \, ds.$$